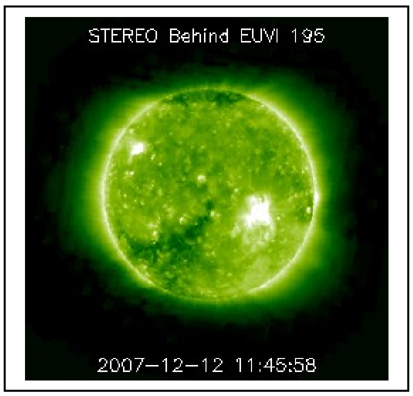
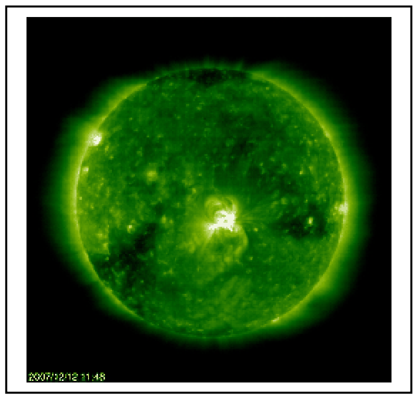
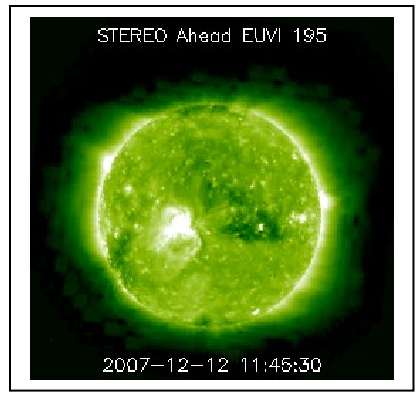
Right Triangle Trigonometry 13.1.1

Two NASA, STEREO satellites take images of the sun and its surroundings from two separate vantage points along Earth's orbit. From these two locations, one located ahead of the Earth, and the other located behind the Earth along its orbit, they can create stereo images of the 3-dimensional locations of coronal mass ejections (CMEs) and storms on or near the solar surface.

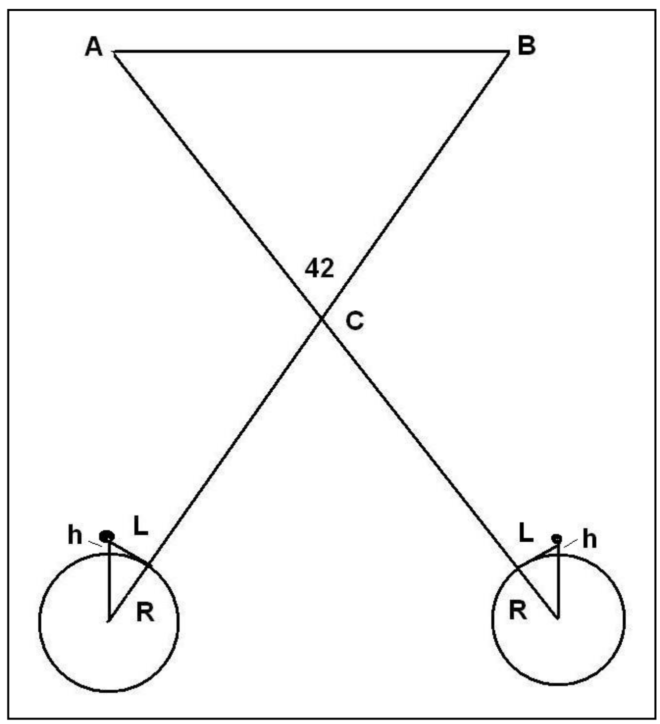
The three images below, taken on December 12, 2007, combine the data from the two STEREO satellites (left and right) taken from these two locations, with the single image taken by the SOHO satellite located half-way between the two STEREO satellites (middle). Notice that there is a large storm event, called Active Region 978, located on the sun. The changing location of AR978 with respect to the SOHO image shows the perspective change seen from the STEREO satellites. You can experience the same *Parallax Effect* by holding your thumb at arms length, and looking at it, first with the left eye, then with the right eye. The location of your thumb will shift in relation to background objects in the room.







The diagram to the right shows the relevant parallax geometry for the two satellites A and B, separated by an angle of 42 degrees as seen from the sun. The diagram lengths are not drawn to scale. The radius of the sun is 696,000 km.



Problem 1: With a millimeter ruler, determine the scale of each image in km/mm. How many kilometers did AR978 shift from the center position (SOHO location for AR) between the two STEREO images? This is the average measure of 'L' in the diagram.

Problem 2: Using the Pythagorean Theorem, determine the equation for the height, h, in terms of R and L. Assume the relevant triangle is a right triangle.

Problem 3: How high (h) above the sun's surface, called the photosphere, was the AR978 viewed by STEREO and SOHO on December 12, 2007?

Answer Key

Problem 1: With a millimeter ruler, determine the scale of each image in km/mm. How many kilometers did AR978 shift from the center position (SOHO location for AR) between the two STEREO images? This is the measure of 'L' in the diagram.

Answer:

STEREO-Left image, sun diameter = 28 mm, actual = 1,392,000 km, so the scale is
1392000 km / 28mm = **49,700 km/mm**
SOHO-center sun diameter = 36 mm, so the scale is
1392000 km/36mm = **38,700 km/mm**
STEREO-right sun diameter = 29 mm, so the scale is
1392000 km / 29 mm = **48,000 km/mm**

Taking the location of the SOHO image for AR978 as the reference, the left-hand image shows that AR978 is about 5 mm to the right of the SOHO location which equals 5 mm x 49,700 km/mm = 248,000 km. From the right-hand STEREO image, we see that AR978 is about 5 mm to the left of the SOHO position or 5 mm x 48,000 km/mm = 240,000 km.
The average is 244,000 kilometers.

Problem 2: Using the Pythagorean Theorem, determine the equation for the height, h, in terms of R and L.

Answer:

$$(R + h)^2 = R^2 + L^2$$

$$h = (L^2 + R^2)^{1/2} - R$$

Problem 3: How high (h) above the sun's surface, called the photosphere, was the AR978 viewed by STEREO and SOHO on December 12, 2007?

Answer:

$$h = ((244{,}000)^2 + (696{,}000)^2)^{1/2} - 696{,}000$$

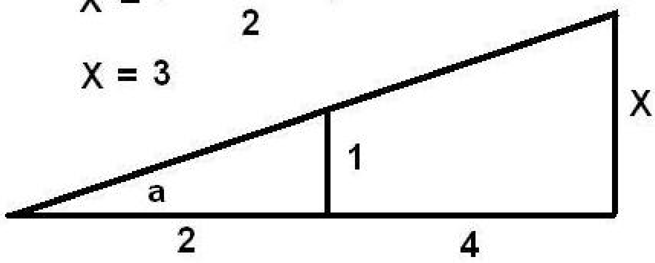
$$h = 737{,}500 - 696{,}000$$

$$h = 41{,}500 \text{ kilometers}$$

$$\frac{X}{(4+2)} = \frac{1}{2}$$

$$X = \frac{(4 + 2) \times 1}{2}$$

$$X = 3$$

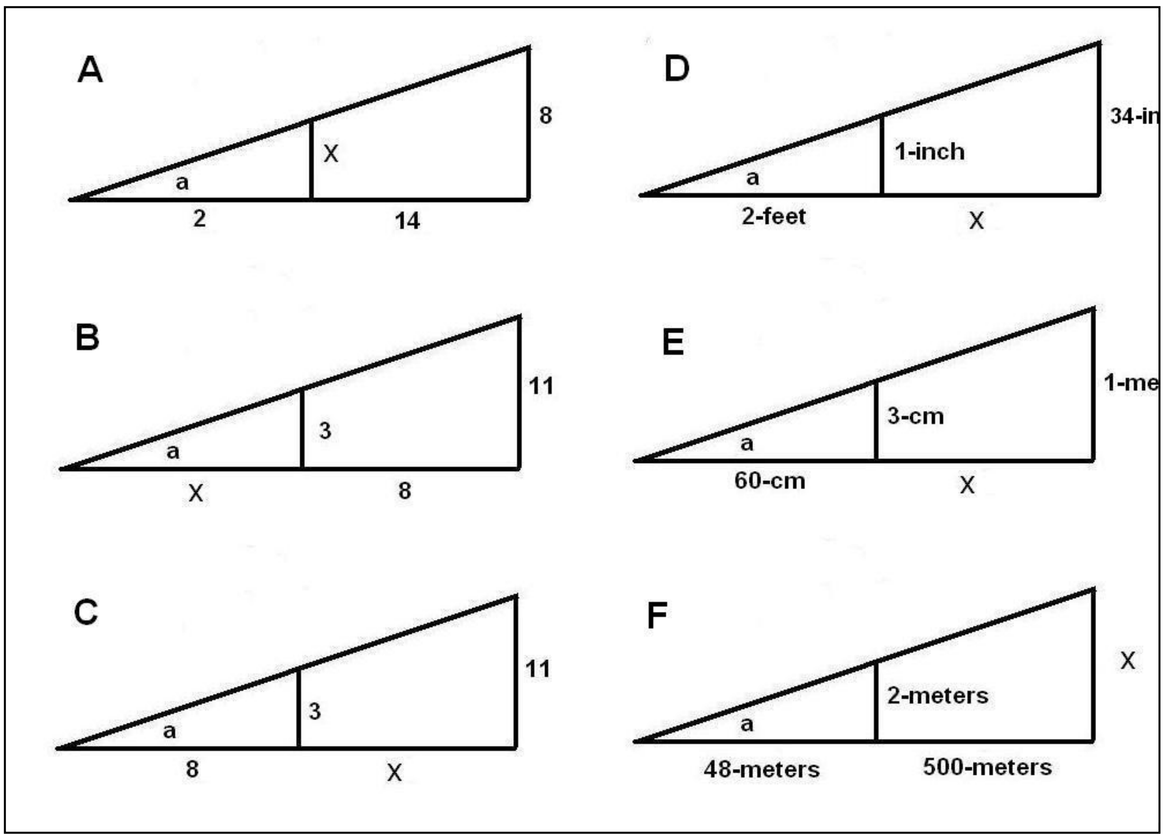


The corresponding sides of similar triangles are proportional to one another as the illustration to the left shows. Because the vertex angle of the triangles are identical in measure, two objects at different distances from the vertex will subtend the same angle, **a** . The corresponding side to 'X' is '1' and the corresponding side to '2' is the combined length of '2+4'.

Problem 1: Use the properties of similar triangles and the ratios of their sides to solve for 'X' in each of the diagrams below.

Problem 2: Which triangles must have the same measure for the indicated angle **a** ?

Problem 3: The Sun is 400 times the diameter of the Moon. Explain why they appear to have the same angular size if the moon is at a distance of 384,000 kilometers?



Answer Key

Problem 1: Use the properties of similar triangles and the ratios of their sides to solve for 'X' in each of the diagrams below.

A) X / 2 = 8 / 16 so **X = 1**

B) 3 / X = 11 / (X+8) so 3 (X + 8) = 11 X ; 3X + 24 = 11 X ; 24 = 8X and so **X = 3**.

C) 3 / 8 = 11 / (x + 8) so 3 (x + 8) = 88 ; 3X + 24 = 88 ; 3X = 64 and so **X = 21 1/3**

D) 1-inch / 24-inches = 34 inches / (X + 24-inches);
so X +24 inches = 34 inches x (24)
and X = 816 - 24 inches and so **X = 792 inches.**

E) 3 cm / 60 cm = 1 meter / (X + 60 cm). 3/60 = 1 meter / (X + 0.6 m) then
3(X + 0.60) = 60 ; 3X + 1.8 = 60 ; 3X = 58.2 meters so **X = 19.4 meters**.

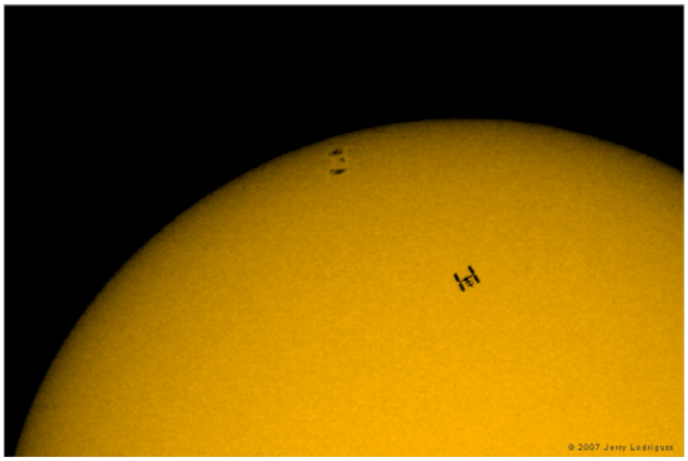
F) 2 meters / 48 meters = X / 548 meters ;
1/24 = X/548 ;
X = 548 / 24 ; so **X = 22.83.**

Problem 2: Which triangles must have the same measure for the indicated angle **a** ?

Answer: Because the triangle (D) has the side proportion 1-inch /24-inches = 1/24 and triangle (F) has the side proportion 2 meters / 48 meters = 1/24 these two triangles**, D and F, have the same angle measurement for angle a.**

Problem 3: The Sun is 400 times the diameter of the Moon. Explain why they appear to have the same angular size if the moon is at a distance of 384,000 kilometers?

Answer: From one of our similar triangles, the long vertical side would represent the diameter of the sun; the short vertical side would represent the diameter of the moon; the angle a is the same for both the sun and moon if the distance to the sun from Earth were 400x farther than the distance of the moon from Earth. Since the lunar distance is 384,000 kilometers, the sun must be at a distance of 154 million kilometers.



The relationship between the distance to an object, **R**, the objects size, **L**, and the angle that it subtends at that distance, θ, is given by:

$$\theta = 57.29 \frac{L}{R} \text{ degrees}$$

$$\theta = 3{,}438 \frac{L}{R} \text{ arcminutes}$$

$$\theta = 206{,}265 \frac{L}{R} \text{ arcseconds}$$

To use these formulae, the units for length, L, and distance, R, must be identical.

Problem 1 - You spot your friend (L = 2 meters) at a distance of 100 meters. What is her angular size in arcminutes?

Problem 2 - The sun is located 150 million kilometers from Earth and has a radius of 696.000 kilometers. What is its angular diameter in arcminutes?

Problem 3 - How far away, in meters, would a dime (1 centimeter) have to be so that its angular size is exactly one arcsecond?

Problem 4 - The spectacular photo above was taken by Jerry Lodriguss (Copyright 2007, http://www.astropix.com/HTML/SHOW_DIG/055.HTM) and shows the International Space Station streaking across the disk of the sun. If the ISS was located 379 kilometers from the camera, and the ISS measured 73 meters across, what was its angular size in arcseconds?

Problem 5 - The orbital speed of the space station is 7.4 kilometers/second. If its distance traveled in 1 second is 7.4 kilometers, A) what was the angle, in arcminutes, that it moved through in one second as seen from the location of the camera? B) What was its angular speed in arcminutes/second?

Problem 6 - Given the diameter of the sun in arcminutes (Problem 2), and the ISS angular speed (Problem 5) how long, in seconds, did it take the ISS to travel across the face of the sun?

Extra for Experts: From the definition of the sine of an angle is given by the formula below, show that for small angles (X << 1), the three formulae above are obtained.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + ...$$

Answer Key

Problem 1 - Answer: Angle = 3,438 x (2 meters/100 meters) = 68.8 arcminutes.

Problem 2 - Answer: 3,438 x (696,000/150 million) = 15.9 arcminutes in radius, so the diameter is 2 x 15.9 = 31.8 arcminutes.

Problem 3 - Answer: From the second formula R = 3438 * L/A = 3438 * 1 cm/1 arcsecond so R = 3,438 centimeters or a distance of 34.4 meters.

Problem 4 - Answer: From the third formula, Angle = 206,265 * (73 meters/379,000 meters) = 39.7 arcseconds.

Problem 5 - Answer: The orbital speed of the space station is 7.4 kilometers/second. If its distance traveled in 1 second is 7.4 kilometers, A) The ISS traveled L = 7.4 kilometers so from the second formula Angle = 3,438 * (7.4 km/379 km) = 67 arcminutes. B) The angular speed is just 67 arcminutes per second.

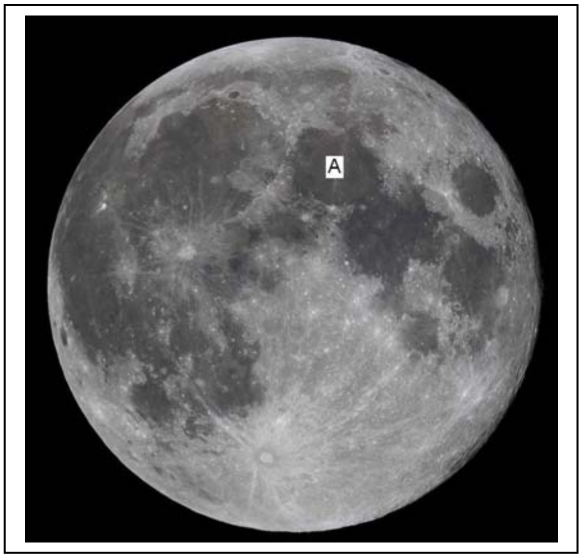
Problem 6 - Answer: The time required is T = 31.8 arcminutes/ (67 arcminutes/sec) = 0.47 seconds.

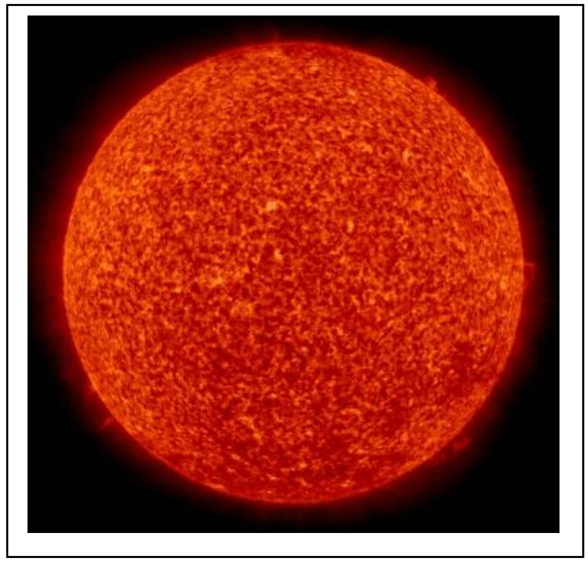
Extra for Experts: From the definition of the sine of an angle is given by the formula below, show that for small angles, the three formulae above are obtained.

Note that as the angle measured in radians, x, becomes very small, only the first term remains significant, so sin (x) = x. Since the length, L, of a chord of a circle with a radius, R, is given by L = R sin(x), we have in the 'small angle approximation', L = R x ,or x = L/R where x is in radians. Since 1 radian = 57.29 degrees, the first formula follows. Since there are 60 arcminutes in 1 arcdegree, 57.2958 x 60 = 3,438 arcminutes in one radian and the second formula follows. Since there are 60 arcseconds in 1 arcminute, 3,438 x 60 = 206,265 arcseconds in one radian and the third formula follows.

The spectacular photo by Jerry Lodriguss had to be taken with careful planning beforehand. He had to know, to the second, when the sun and ISS would be in the right configuration in the sky as viewed from his exact geographic location. Here's an example of the photography considerations in his own words:

" I considered trying to monitor the transit visually with a remote release in my hand and just firing (the camera) when I saw the ISS in my guidescope. Then I worked out the numbers. I know that my reaction time is 0.19 seconds. This is actually quite good, but I make my living shooting sports where this is critical, so I better be good at it. I also know that the Canon 1D Mark IIn has a shutter lag of 55 milliseconds. Adding these together, plus a little bit of a fudge factor, the best I could hope for was about 1/4 of a second from when I saw it to when the shutter opened. Since the entire duration of the transit was only 1/2 of a second, in theory, I could capture the ISS at about the center of the disk if I fired as soon as I saw it start to cross. This was not much of a margin for error. I could easily blink and miss the whole thing... Out of 49 frames that the Mark IIn recorded, the ISS is visible in exactly one frame."





The Sun (Diameter = 696,000 km) and Moon (Diameter = 3,476 km) have very different physical diameters in kilometers, but in the sky they can appear to be nearly the same size. Astronomers use the angular measure of arcseconds (asec) to measure the apparent sizes of most astronomical objects. (1 degree equals 60 arcminutes, and 1 arcminute equals 60 arcseconds). The photos above show the Sun and Moon at a time when their angular diameters were both about 1,865 arcseconds.

Problem 1 - Using a metric ruler, what is the angular scale of each image in arcseconds per millimeter?

Problem 2 - In arcseconds, what is the size of the smallest feature you can see in the images of the Sun and Moon?

Problem 3 - About what is the area, in square arcseconds ($asec^2$) of the circular Mare Serenitatis (A) region in the photo of the Moon?

Problem 4 - At the distance of the Moon, 1 arcsecond of angular measure equals 1.9 kilometers. The Sun is exactly 400 times farther away than the Moon. On the photograph of the Sun, how many kilometers equals 1 arcsecond of angle?

Problem 5 - What is the area of Mare Serenitatis in square kilometers?

Problem 6 - What would be the physical area, in square-kilometers, of an identical angular area to Mare Serenitatis if it were located on the surface of the sun?

Answer Key

Problem 1 - Using a metric ruler, what is the angular scale of each image in arcseconds per millimeter? Answer: Moon diameter = 65 mm and sun diameter = 61 mm so the lunar image scale is 1,865 asec/65mm **= 28.7 asec/mm** and the solar scale is 1865 asec/61 mm = **30.6 asec/mm.**

Problem 2 - In arcseconds, what is the size of the smallest feature you can see in the images of the Sun and Moon? Answer: the smallest feature is about 0.5 mm or 0.5 x 28.7 asec/mm = **14.4 asec for the Moon** and 0.5 x 30.6 asec/mm = **15.3 asec for the Sun**.

Problem 3 - About what is the area, in square arcseconds ($asec^2$) of the circular Mare Serenitatis (A) region in the photo of the Moon? Answer: The diameter of the mare is 1 centimeter, so the radius is 5 mm or 5 mm x 28.7 asec/mm = 143.5 asec. Assuming a circle, the area is $A = \pi \times (143.5 \text{ asec})^2$ = **64,700 $asec^2$.**

Problem 4 - At the distance of the Moon, 1 arcsecond of angular measure equals 1.9 kilometers. The Sun is exactly 400 times farther away than the Moon. On the photograph of the Sun, how many kilometers equals 1 arcsecond of angle? Answer: The angular scale at the sun would correspond to 400 x 1.9 km = **760 kilometers per arcsecond.**

Problem 5 - What is the area of Mare Serenitatis in square kilometers? Answer: We have to convert from square arcseconds to square kilometers using a two-step unit conversion 'ladder'.

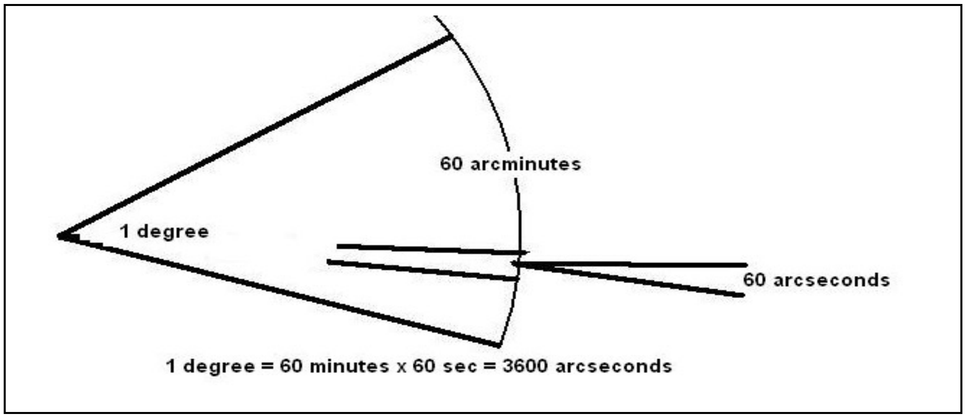
$$64{,}700 \text{ asec}^2 \times (1.9 \text{ km/asec}) \times (1.9 \text{ km/asec}) = \mathbf{233{,}600 \text{ km}^2.}$$

Problem 6 - What would be the physical area, in square-kilometers, of an identical angular area to Mare Serenitatis if it were located on the surface of the sun? Answer: The angular area is 400-times further away, so we have to use the scaling of 760 kilometers/asec deduced in Problem 4. The unit conversion for the solar area becomes:

$$64{,}700 \text{ asec}^2 \times (760 \text{ km/asec}) \times (760 \text{ km/asec}) = \mathbf{37{,}400{,}000{,}000 \text{ km}^2.}$$

General Angles and Radian Measure

The easiest, and most basic, unit of measure in astronomy is the angular degree. Because the distances to objects in the sky are not directly measurable, a photograph of the sky will only indicate how large, or far apart, objects are in terms of degrees, or fractions of degrees. It is a basic fact in angle mensuration in geometry, that 1 angular degree (or arc-degree) can be split into 60 arc-minutes of angle, and that 1 arc-minute equals 60 arc-seconds. A full degree is then equal to 60 x 60 = 3,600 'arcseconds'. High-precision astronomy also uses the unit of milliarcsecond to represent angles as small as 0.001 arcseconds and microarcseconds to equal 0.000001 arcseconds.



Problem 1 – The moon has a diameter of 0.5 degrees (a physical size of 3,474 km) A telescope sees a crater 1 arcsecond across. What is its diameter in meters?

Problem 2 – A photograph has an image scale of 10 arcseconds/pixel. If the image has a size of 512 x 512 pixels, what is the image field-of-view in degrees?

Problem 3 – An astronomer wants to photograph the Orion Nebula (M-42) with an electronic camera with a CCD format of 4096x4096 pixels. If the nebula has a diameter of 85 arcminutes. What is the resolution of the camera in arcseconds/pixel when the nebula fills the entire field-of-view?

Problem 4 – An electronic camera is used to photograph the Whirlpool Galaxy, M-51, which has a diameter of 11.2 arcminutes. The image will have 1024x1024 pixels. What is the resolution of the camera, in arcseconds/pixel, when the galaxy fills the entire field-of-view?

Problem 5 – The angular diameter of Mars from Earth is about 25 arcseconds. This corresponds to a linear size of 6,800 km. The Mars Reconnaissance Orbiter's HiRISE camera, in orbit around Mars, can see details as small as 1 meter. What is the angular resolution of the camera in microarcseconds as viewed from Earth?

Problem 6 – The Hubble Space Telescope can resolve details as small as 46 milliarcseconds. At the distance of the Moon, how large a crater could it resolve, in meters?

Answer Key

Problem 1 – Answer: 0.5 degrees x 3600 arcsec/degree = 1800 arcseconds. Using proportions 1/1800 = x/3474 so X = 3474/1800 = 1.9 kilometers.

Problem 2 –Answer: 512 pixels x 10 arcsec/pixel x 1 degree/3600 arcseconds = 5120 arcseconds /3600 = 1.4 degrees, so the image is 1.4 x 1.4 degrees.

Problem 3 –Answer: 85 arcminutes x 60 arcsec/arcmin = 5,100 arcseconds. This corresponds to 4096 pixels so the scale is 5,100 arcsec/4096 pixels = 1.2 arcsec/pixel.

Problem 4 –Answer: 11.2 arcminutes x 60 arcsec/arcmin = 672 arcsec. This equals 1024 opixels so the scale is 672/1024 = 0.656 arcsec/pixel.

Problem 5 –Answer: 25 arcsec = 6800 km so 1 arcsec = 6800 km/25 = 272 km from Earth. For 1-meter resolution at Earth, the angular scale would have to be 1 sec x 1m/272000m = 0.0000037 arcseconds or 3.7 microarcseconds.

Problem 6 – Answer: From Problem-1, 1 arcsecond = 1.9 kilometers. By proportions, 0.046 arcsec/1 arcsec = x/1.9 km so X = 0.046 x 1.9 km = 0.087 kilometers or 87 meters.

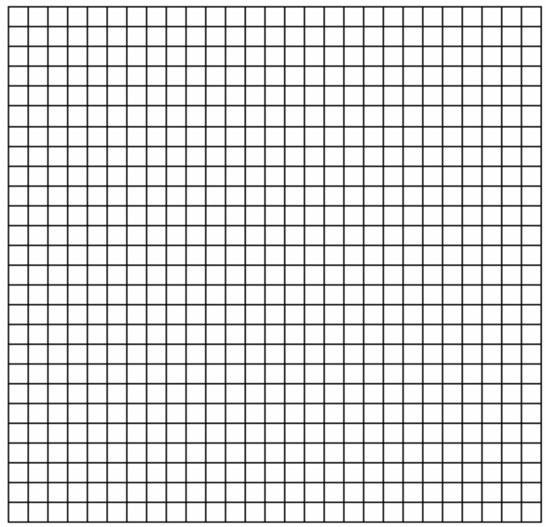
The picture below was taken by the Cassini spacecraft orbiting Saturn. It is of the satellite Phoebe, which from Earth subtends an angular size of about 32 milliarcsec. The smallest crater, about 1 km across, would subtend about 160 microarcseconds as seen from Earth.

General Angles and Radian Measure 13.2.4

Although a pair of binoculars or a telescope can see amazing details on the Moon, the human eye is not so gifted!

The lens of the eye is so small, only 2 to 5 millimeters across, that the sky is 'pixelized' into cells that are about one arcminute across. We call this the resolution limit of the eye, or the eye's visual acuity.

One degree of angle measure can be divided into 60 minutes of arc. For an object like the full moon, which is 1/2-degree in diameter, it also measures 30 arcminutes in diameter. This means that, compared to the human eye, the moon can be divided into an image that is 30-pixels in diameter.



Problem 1 - Convert the following degree measures into their equivalent measure in arcminutes; A) 5 degrees; B) 2/3 degree; C) 15.5 degrees; D) 0.25 degrees

Problem 2 - Convert the following arcminute measures into their equivalent measure in degrees: A) 15 amin; B) 1/2 amin; C) 120.5 amin; D) 3600 amin.

Problem 3 - Convert the following area measures in square-degrees into their equivalent measures in square arcminutes ($amin^2$): A) 1.0 deg^2; B) 0.25 deg^2

Problem 4 - The figure to the above-left is a telescopic photo of the full moon showing its many details including craters and dark mare. Construct a simulated image of the moon in the grid to the right to represent what the moon would look like at the resolution of the human eye. First sketch the moon on the grid. Then use the three shades; black, light-gray and dark-gray, and fill-in each square with one of the three shades using your sketch as a guide.

Problem 5 - Why can't the human eye see any craters on the Moon?

Answer Key

Problem 1 - Convert the following degree measures into their equivalent measure in arcminutes; A) 5 degrees; B) 2/3 degree; C) 15.5 degrees; D) 0.25 degrees

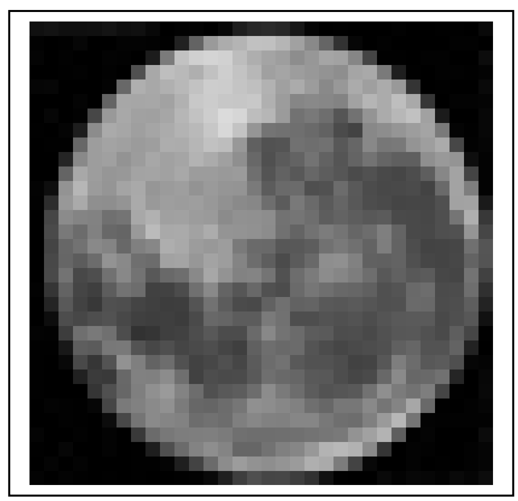
Answer: A) 5 degrees x (60 amin/1 deg) = **300 amin**. B) 2/3 degree x (60 amin/1 deg) = 120/3 = **40 amin**. C) 15.5 degrees x (60 amin/1deg) = **930 amin**; D) 0.25 deg x (60 amin/1 deg) = **15 amin**.

Problem 2 - Convert the following arcminute measures into their equivalent measure in degrees: A) 15 amin; B) 1/2 amin; C) 120.5 amin; D) 3600 amin.
Answer: A) 15 amin x (1 deg/60 amin) = **0.25 deg**. B) 1/2 amin x (1 deg / 60 amin) = **1/120 deg**. C) 120.5 amin x (1 deg/60 amin) = **2.008 deg.** D) 360 amin x (1 deg/60 amin) = **60 deg.**

Problem 3 - Convert the following area measures in square-degrees into their equivalent measures in square arcminutes ($amin^2$): A) 1.0 deg^2; B) 0.25 deg^2
Answer; A) 1.0 deg^2 x (60 amin/1 deg) x (60 amin/1 deg) = **3600 $amin^2$**. B) 0.25 deg^2 x (60 amin/1 deg) x (60 amin/1 deg) = 0.25 x 3600 = **900 $amin^2$.**

Problem 4 - See the image below which has been pixelized to the grid resolution. How well did your version match the image on the right?

Problem 5 - Why can't the human eye see any craters on the Moon? Answer: The human eye can only see details 1 arcminute across and this is too low a resolution to see even the largest craters.



When astronomers take photographs of a specific region of the sky, they often take hundreds of separate images and then average them together to increase the sensitivity of the image.

Each image can be shifted and rotated with respect to the previous image, so these changes have to be determined, mathematically, and removed so that the images can be combined.

Two digital images were taken by an astronomer of the same star field at two separate days. The goal is to look for stars whose brightnesses have suddenly dimmed in order to detect planets passing across the disk of the star as viewed from Earth. In each image, the astronomer can identify four stars and measure their 'X-Y' locations on the image, where the units of X and Y are in pixels.

Problem 1 - From the table below, what are the polar coordinates of the stars in Image A?

Problem 2 – The astronomer determines that there were two possible rotation angles for Image B. It could either have been rotated clockwise by 36^{o} or clockwise by 18^{o} with respect to Image A. By what angle was Image B rotated with respect to Star C?

	Image A (x,y)	Image B (x,y)
Star A	(+327, +843)	(+492, +757)
Star B	(-193, -50)	(-185, +75)
Star C	(0,0)	(0,0)
Star D	(-217, +33)	(-155, +155)

Problem 3 – If the angular distance between Star A and Star C is 12.8 degrees, how far apart are the other two stars from Star C measured in degrees?

Problem 1 - From the table below, what are the polar coordinates of the stars in Image A?

	Image A (x,y)	Image B (x,y)
Star A	(+327, +843)	(+492, +757)
Star B	(-193, -50)	(-185, +75)
Star C	(0,0)	(0,0)
Star D	(-217, +33)	(-155, +155)

Answer: Using Star C as the origin:

Star A: $R^2 = (327)^2 + (843)^2$ so R = 903 and sin θ = 843/903 so θ = 69^o
Star B: $R^2 = (193)^2 + (50)^2$ so R = 199 and sin θ = 50/199 so θ = 180+15 = 195^o
Star D: $R^2 = (217)^2 + (33)^2$ so R = 219 and sin θ = 33/219 so θ = 180-9 = 171^o

So **Star A: (903, +69^o) Star B: (199, +195^o) Star D: (219, +171^o)**

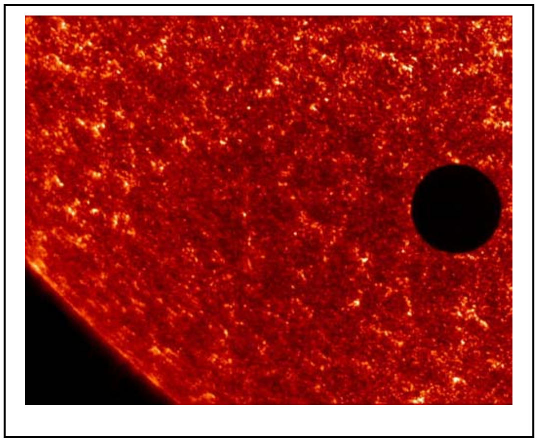
Problem 2 - The astronomer determines that there were two possible rotation angles for Image B. It could either have been rotated clockwise by 36^o or clockwise by 18^o with respect to Image A. By what angle was Image B rotated with respect to Star C?

Answer: Since all stars will move in unison as they rotate, we only need to test for the correct solution using one of the star coordinates: Select Star A:
Since x = R cos(θ) and y = Rsin(θ) then
For 36^o clockwise we have θ = +69^o – 36^o = 33^o so
X = 903 cos(33) = +757 and y = 903 sin(33) = +492 which match the Image B coordinates. So θ **= 36^o is the correct rotation angle.**

Problem 2 – If the angular distance between Star A and Star C is 12.8 degrees, how far apart are the other two stars from Star C measured in degrees?

Answer: We know that, in pixel units, Star A-C = 903, so the scale of the image is just 12.8 degrees/903 pixels or 0.014 degrees/pixel. The radial coordinates for the stars are then:
Star A : 12.8 degrees
Star B : 199 pixels x (0.014 degrees/pixel) = **2.8 degrees**
Star D: 219 pixels x (0.014 degrees/pixel) = **3.1 degrees**

The Law of Sines



On June 5, 2012 the planet Venus will pass across the face of the sun as viewed from Earth. The last time this happened was on June 6, 2004. A similar event, called the Transit of Venus' will not happen again until December 11, 2117.

This image, taken by the TRACE satellite, shows the black disk of Venus passing across the solar disk photographed with a filter that highlights details on the solar surface.

Problem 1 – The table below gives the location of the other 7 planets at the time of the 2012 Transit of Venus, with the sun at the origin of the Cartesian coordinate system. What are the polar coordinates for each planet at the time of the next transit? Note: All distances are in terms of the Astronomical Unit (AU) for which the distance between the Sun and Earth equals exactly 1.0 AU.

Planet	X	Y	R	θ
Sun	0.000	0.000		
Mercury	-0.157	+0.279		
Venus	-0.200	-0.698		
Earth	-0.268	-0.979		
Mars	-1.450	-0.694		
Jupiter	+2.871	+4.098		
Saturn	-8.543	-4.635		
Uranus	+19.922	+1.988		
Neptune	+26.272	-14.355		

Problem 2 – In the sky as viewed from Earth, what is the angular distance between the Sun and Mars if the angle between the Sun and Earth as viewed from Mars is 39^{o}?

Problem 1 – The table below gives the location of the other 7 planets at the time of the 2012 Transit of Venus, with the sun at the origin of the Cartesian coordinate system. What are the polar coordinates for each planet at the time of the next transit? All angles are measured in positive direction counterclockwise with respect to the + X axis. (Note: All distances are in terms of the Astronomical Unit (AU) for which the distance between the Sun and Earth equals exactly 1.0 AU.)

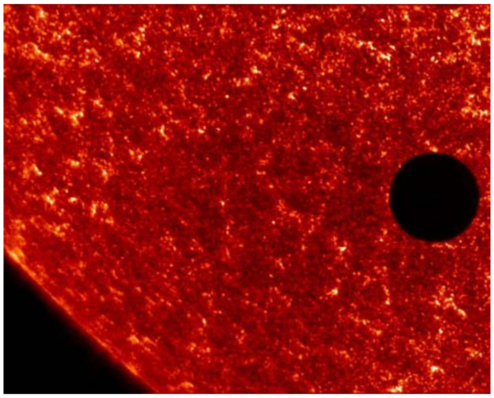
Planet	X	Y	R	θ
Sun	0.000	0.000		
Mercury	-0.157	+0.279	0.320	119
Venus	-0.200	-0.698	0.726	254
Earth	-0.268	-0.979	1.015	255
Mars	-1.450	-0.694	1.608	206
Jupiter	+2.871	+4.098	5.003	55
Saturn	-8.543	-4.635	9.719	208
Uranus	+19.922	+1.988	20.021	6
Neptune	+26.272	-14.355	29.938	331

Answer: Example for Jupiter: $R^2 = (2.871)^2 + (4.098)^2$ so R = 5.003 AU.
The angle can be determined from $\cos(\theta) = X/R$, for Earth, located in the Third Quadrant: $\cos^{-1}(-0.268/1.015) = 105^o$ so $360^o - 105^o = 255^o$.

Problem 2 – In the sky as viewed from Earth, what is the angular distance between the Sun and Mars if the angle between the Sun and Earth as viewed from Mars is 39^o?

Answer: Sun-Earth-Mars form a triangle with the Earth at one vertex. You need to calculate this vertex angle. Use the Law of Sines:

$$\frac{\text{Sin}(39)}{1.015} = \frac{\sin(x)}{1.608}$$ so $\sin(x) = 0.9970$ and so **x = 86 degrees**.



On June 5, 2012 the planet Venus will pass across the face of the sun as viewed from Earth. The last time this happened was on June 6, 2004. A similar event, called the Transit of Venus' will not happen again until December 11, 2117.

This image, taken by the TRACE satellite, shows the black disk of Venus passing across the solar disk photographed with a filter that highlights details on the solar surface.

Problem 1 – The table below gives the location of the other 7 planets at the time of the 2012 Transit of Venus, with the sun at the origin of the Cartesian coordinate system. What are the polar coordinates for each planet at the time of the next transit? Note: All distances are in terms of the Astronomical Unit (AU) for which the distance between the Sun and Earth equals exactly 1.0 AU.

Planet	X	Y	R	θ
Sun	0.000	0.000		
Mercury	-0.157	+0.279		
Venus	-0.200	-0.698		
Earth	-0.268	-0.979		
Mars	-1.450	-0.694		
Jupiter	+2.871	+4.098		
Saturn	-8.543	-4.635		
Uranus	+19.922	+1.988		
Neptune	+26.272	-14.355		

Problem 2 – In the sky as viewed from Earth, what is the angular distance between the Sun and Mars?

Problem 1 – The table below gives the location of the other 7 planets at the time of the 2012 Transit of Venus, with the sun at the origin of the Cartesian coordinate system. What are the polar coordinates for each planet at the time of the next transit? All angles are measured in positive direction counterclockwise with respect to the + X axis. (Note: All distances are in terms of the Astronomical Unit (AU) for which the distance between the Sun and Earth equals exactly 1.0 AU.)

Planet	X	Y	R	θ
Sun	0.000	0.000		
Mercury	-0.157	+0.279	0.320	119
Venus	-0.200	-0.698	0.726	254
Earth	-0.268	-0.979	1.015	255
Mars	-1.450	-0.694	1.608	206
Jupiter	+2.871	+4.098	5.003	55
Saturn	-8.543	-4.635	9.719	208
Uranus	+19.922	+1.988	20.021	6
Neptune	+26.272	-14.355	29.938	331

Answer: Example for Jupiter: $R^2 = (2.871)^2 + (4.098)^2$ so R = 5.003 AU.
The angle can be determined from $\cos(\theta) = X/R$, for Earth, located in the Third Quadrant: $\cos^{-1}(-0.268/1.015) = 105^o$ so $360^o - 105^o = 255^o$.

Problem 2 – In the sky as viewed from Earth, what is the angular distance between the Sun and Mars?

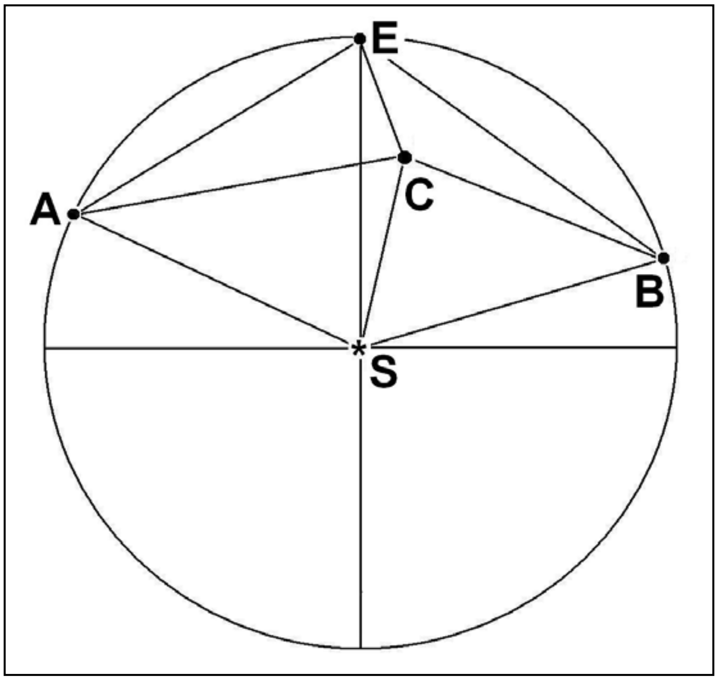
Answer: Sun-Earth-Mars form a triangle with the Earth at one vertex. You need to calculate this vertex angle. Use the Law of Cosines:

$$L^2 = a^2 + b^2 - 2ab\cos A$$

Where L = Mars-Sun distance = 1.608 AU
A = Earth-Sun distance = 1.015 AU
B = Earth-Mars distance: $B^2 = (-0.269 + 1.450)^2 + (-0.979 + 0.694)^2$ so B = 1.215 AU

Then CosA = 0.032 so **A = 88 degrees**.

Law of Cosines



The two STEREO spacecraft are located along Earth's orbit and can view gas clouds ejected by the sun as they travel to Earth. From the geometry, astronomers can accurately determine their speeds, distances, shapes and other properties.

By studying the separate 'stereo' images, astronomers can determine the speed and direction of the cloud before it reaches Earth.

Use the diagram, (angles and distances not drawn to the same scale of the 'givens' below) to answer the following question.

The two STEREO satellites are located at points A and B, with Earth located at Point E and the sun located at Point S, which is the center of a circle with a radius ES of 1.0 Astronomical unit (150 million kilometers). Suppose that the two satellites spot a Coronal Mass Ejection (CME) cloud at Point C. Satellite A measures its angle from the sun mSAC as 45 degrees while Satellite B measures the corresponding angle to be mSBC=50 degrees. In the previous math problem the astronomers knew the ejection angle of the CME, mESC, but in fact they didn't need to know this in order to solve the problem below!

Problem 1 - The astronomers want to know the distance that the CME is from Earth, which is the length of the segment EC. The also want to know the approach angle, mSEC. Use either a scaled construction (easy: using compass, protractor and millimeter ruler) or geometric calculation (difficult: using trigonometric identities) to determine EC from the available data.

Givens from satellite orbits:

SB = SA = SE = 150 million km AE = 136 million km BE = 122 million km
mASE = 54 degrees mBSE = 48 degrees
mEAS = 63 degrees mEBS = 66 degrees mAEB = 129 degrees

Find the measures of all of the angles and segment lengths in thee above diagram rounded to the nearest integer.

Problem 2 - If the CME was traveling at 2 million km/hour, how long did it take to reach the distance indicated by the length of segment SC?

Givens from satellite orbits:

SB = SA = SE = 150 million km   AE = 136 million km   BE = 122 million km
mASE = 54 degrees   mBSE = 48 degrees
mEAS = 63 degrees   mEBS = 66 degrees   mAEB = 129 degrees
use units of megakilometers i.e. 150 million km = 150 Mkm.

Method 1: Students construct a scaled model of the diagram based on the angles and measures, then use a protractor to measure the missing angles, and from the scale of the figure (in millions of kilometers per millimeter) they can measure the required segments. **Segment EC is about 49 Mkm at an angle, mSEB of 28 degrees.**

Method 2 use the Law of Cosines and the Law of Sines to solve for the angles and segment lengths exactly.

mASB = mASE+mBSE = 102 degrees
mASC = theta
mACS = 360 - mCAS - mASC = 315 - theta
mBSC = mASB - theta = 102 - theta
mBCS = 360 - mCBS - mBSC = 208 + theta

Use the Law of Sines to get
sin(mCAS)/L = sin(mACS)/150 Mkm   and   sin(mBCS)/L = sin(mBCS)/150 Mkm.

Eliminate L :   150sin(45)/sin(315-Theta) = 150 sin(50)/sin(208+theta)

Re-write using angle-addition and angle-subtraction:
sin50 [sin(315)cos(theta)-cos(315)sin(theta)] = sin(45) [sin(208)cos(theta) + cos(208)sin(theta)]

Compute numerical factors by taking indicated sines and cosines:
-0.541cos(theta) - 0.541sin(theta) = -0.332cos(theta) - 0.624sin(theta)

Simplify: cos(theta) = 0.397sin(theta)

Use definition of sine: $\cos(\theta)^2 = 0.158 (1 - \cos(\theta)^2)$

Solve for cosine: $\cos(\theta) = (0.158/1.158)^{1/2}$ so theta = 68 degrees. And so **mASC=68**

Now compute segment CS = 150 sin(45)/sin(315-68) = **115 Mkm**.
BC = 115 sin(102-68)/sin(50) = **84 Mkm**.
Then $EC^2 = 122^2 + 84^2 - 2(84)(122)\cos(mEBS-mCBS)$
$EC^2 = 122^2 + 84^2 - 2(84)(122)\cos(66-50)$ So **EC = 47 Mkm.**

mCEB from Law of Cosines: $84^2 = 122^2 + 47^2 - 2(122)(47)\cos(mCEB)$ so **mCEB = 29 degrees**

And since   mAES = 180 - mASE - mEAS = 180 - 54 - 63 = 63 degrees
so mSEC = mAEB - mAES - mCEB = 129 - 63 - 29 = **37 degrees**

So, the two satellites are able to determine that the CME is 49 million kilometers from Earth and approaching at an angle of 37 degrees from the sun.

Problem 2 - If the CME was traveling at 2 million km/hour, how long did it take to reach the distance indicated by the length of segment SC?
Answer: 115 million kilometers / 2 million km/hr = **58 hours or 2.4 days.**

Parametric Equations

The neat thing about ballistic problems (flying baseballs or rockets) is that their motion in the vertical dimension is independent of their motion in the horizontal dimension. This means we can write one equation that describes the movement in time along the x axis, and a second equation that describes the movement in time along the y axis. In function notation, we write these as x(t) and y(t) where t is the independent variable representing time.

To draw the curve representing the trajectory, we have a choice to make. We can either create a table for X and Y at various instants in time, or we can simply eliminate the independent variable, t, and plot the curve y(x).

Problem 1 - The Ares 1X underwent powered flight while its first stage rocket engines were operating, but after it reached the highest point in its trajectory (apogee) the Ares 1X capsule coasted back to Earth for a water landing. The parametric equations that defined its horizontal downrange location (x) and its altitude (y) in meters are given by

$$x(t) = 64,000 + 1800t$$
$$y(t) = 45,000 - 4.9t^2$$

Using the method of substitution, create the new function y(x) by eliminating the variable t.

Problem 2 - Determine how far downrange from launch pad 39A at Cape Canaveral the capsule landed, (y(x)=0), giving your answer in both meters and kilometers to two significant figures.

Problem 3 - Why is it sometimes easier not to work with the parametric form of the motion of a rocket or projectile?

Problem 1 - Answer: The parametric equations that defined its horizontal downrange location (x) and its altitude (y) in meters are given by

$$x(t) = 64,000 + 1800t$$
$$y(t) = 45,000 - 4.9t^2$$

We want to eliminate the variable, t, from y(t) and we do this by solving the equation x(t) for t and substituting this into the equation for y(t) to get y(t(x)) or just y(x).

$$t = \frac{x - 64,000}{1800}$$

$$y = 45,000 - 4.9\left[\frac{x - 64,000}{1800}\right]^2$$

$$y = 38,800 + 0.19x - 0.0000015x^2$$

Problem 2 - Determine how far downrange from the launch pad the capsule landed, (y(x)=0), giving your answer in both meters and kilometers to two significant figures. Answer: Using the Quadratic Formula, find the two roots of the equation y(x)=0, and select the root with the largest positive value.

$$x_{1,2} = \frac{-0.19 \pm \sqrt{0.036 - 4(-0.0000015)38800}}{2(-0.0000015)} \text{ meters}$$

so x1 = -109,000 meters or -109 kilometers
x2 = **+237,000 meters or 237 kilometers.**

The second root, x2, is the answer that is physically consistent with the given information. Students may wonder why the mathematical model gives a second answer of -109 kilometers. This is because the parabolic model was only designed to accurately represent the physical circumstances of the coasting phase of the capsule's descent from its apogee at a distance of 64 kilometers from the launch pad. Any extrapolations to times and positions earlier than the moment of apogee are 'unphysical'.

Problem 3 - Why is it sometimes easier not to work with the parametric form of the motion of a rocket or projectile? Answer: In order to determine the trajectory in space, you need to make twice as many calculations for the parametric form than for the functional form y(x) since each point is defined by (x(t), y(t)) vs (x, y(x)).